

10.4 Inverse of a Function

Essential Question How are a function and its inverse related?

EXPLORATION 1 Exploring Inverse Functions

Work with a partner. The functions f and g are *inverses* of each other. Compare the tables of values of the two functions. How are the functions related?

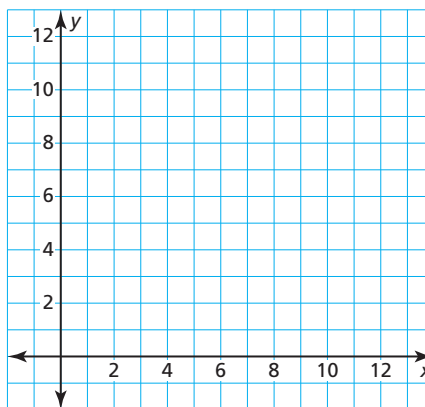
x	0	0.5	1	1.5	2	2.5	3	3.5
$f(x)$	0	0.25	1	2.25	4	6.25	9	12.25

x	0	0.25	1	2.25	4	6.25	9	12.25
$g(x)$	0	0.5	1	1.5	2	2.5	3	3.5

EXPLORATION 2 Exploring Inverse Functions

Work with a partner.

- Plot the two sets of points represented by the tables in Exploration 1. Use the coordinate plane below.
- Connect each set of points with a smooth curve.
- Describe the relationship between the two graphs.
- Write an equation for each function.



Communicate Your Answer

- How are a function and its inverse related?
- A table of values for a function f is given. Create a table of values for a function g , the inverse of f .

x	0	1	2	3	4	5	6	7
$f(x)$	1	2	3	4	5	6	7	8

- Sketch the graphs of $f(x) = x + 4$ and its inverse in the same coordinate plane. Then write an equation of the inverse of f . Explain your reasoning.

ATTENDING TO PRECISION

To be proficient in math, you need to communicate precisely with others.



10.4 Lesson

Core Vocabulary

inverse relation, p. 568
inverse function, p. 569

Previous

input
output
inverse operations
reflection
line of reflection

What You Will Learn

- ▶ Find inverses of relations.
- ▶ Explore inverses of functions.
- ▶ Find inverses of functions algebraically.
- ▶ Find inverses of nonlinear functions.

Finding Inverses of Relations

Recall that a relation pairs inputs with outputs. An **inverse relation** switches the input and output values of the original relation.

Core Concept

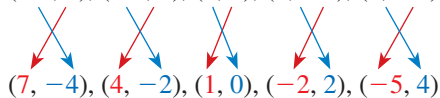
Inverse Relation

When a relation contains (a, b) , the inverse relation contains (b, a) .

EXAMPLE 1 Finding Inverses of Relations

Find the inverse of each relation.

- a. $(-4, 7), (-2, 4), (0, 1), (2, -2), (4, -5)$



Switch the coordinates of each ordered pair.

Inverse relation

b.

Input	-1	0	1	2	3	4
Output	5	10	15	20	25	30

Inverse relation:

Input	5	10	15	20	25	30
Output	-1	0	1	2	3	4

Switch the inputs and outputs.

Monitoring Progress



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Find the inverse of the relation.

1. $(-3, -4), (-2, 0), (-1, 4), (0, 8), (1, 12), (2, 16), (3, 20)$

2.

Input	-2	-1	0	1	2
Output	4	1	0	1	4

Exploring Inverses of Functions

Throughout this book, you have used given inputs to find corresponding outputs of $y = f(x)$ for various types of functions. You have also used given outputs to find corresponding inputs. Now you will solve equations of the form $y = f(x)$ for x to obtain a formula for finding the input given a specific output of the function f .

EXAMPLE 2

Writing a Formula for the Input of a Function

Let $f(x) = 2x + 1$. Solve $y = f(x)$ for x . Then find the input when the output is -3 .

SOLUTION

$$y = 2x + 1 \quad \text{Set } y \text{ equal to } f(x).$$

$$y - 1 = 2x \quad \text{Subtract 1 from each side.}$$

$$\frac{y - 1}{2} = x \quad \text{Divide each side by 2.}$$

Find the input when $y = -3$.

$$x = \frac{-3 - 1}{2} \quad \text{Substitute } -3 \text{ for } y.$$

$$= \frac{-4}{2} \quad \text{Subtract.}$$

$$= -2 \quad \text{Divide.}$$

Check

$$\begin{aligned} f(-2) &= 2(-2) + 1 \\ &= -4 + 1 \\ &= -3 \quad \checkmark \end{aligned}$$

► So, the input is -2 when the output is -3 .

Monitoring Progress



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Solve $y = f(x)$ for x . Then find the input when the output is 4.

3. $f(x) = x - 6$

4. $f(x) = \frac{1}{2}x + 3$

5. $f(x) = 4x^2$

UNDERSTANDING MATHEMATICAL TERMS

The term *inverse functions* does not refer to a new type of function. Rather, it describes any pair of functions that are inverses.

In Example 2, notice the steps involved after substituting for x in $y = 2x + 1$ and after substituting for y in $x = \frac{y - 1}{2}$.

$$y = 2x + 1$$

$$x = \frac{y - 1}{2}$$

Step 1 Multiply by 2.

Step 1 Subtract 1.

Step 2 Add 1.

Step 2 Divide by 2.

inverse operations in the reverse order

Notice that these steps *undo* each other. **Inverse functions** are functions that undo each other. In Example 2, you can use the equation solved for x to write the inverse of f by switching the roles of x and y .

$$f(x) = 2x + 1 \quad \text{original function}$$

$$g(x) = \frac{x - 1}{2} \quad \text{inverse function}$$

Because an inverse function interchanges the input and output values of the original function, the domain and range are also interchanged.

LOOKING FOR A PATTERN

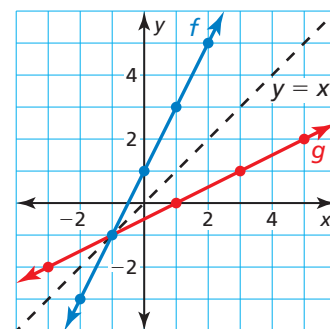
Notice that the graph of the inverse function g is a reflection of the graph of the original function f . The line of reflection is $y = x$.

Original function: $f(x) = 2x + 1$

x	-2	-1	0	1	2
y	-3	-1	1	3	5

Inverse function: $g(x) = \frac{x - 1}{2}$

x	-3	-1	1	3	5
y	-2	-1	0	1	2



Finding Inverses of Functions Algebraically

Core Concept

Finding Inverses of Functions Algebraically

- Step 1** Set y equal to $f(x)$.
Step 2 Switch x and y in the equation.
Step 3 Solve the equation for y .

STUDY TIP

On the previous page, you solved a function for x and switched the roles of x and y to find the inverse function. You can also find the inverse function by switching x and y first, and then solving for y .

EXAMPLE 3 Finding the Inverse of a Linear Function

Find the inverse of $f(x) = 4x - 9$.

SOLUTION

Method 1 Use the method above.

Step 1 $f(x) = 4x - 9$ Write the function.

$y = 4x - 9$ Set y equal to $f(x)$.

Step 2 $x = 4y - 9$ Switch x and y in the equation.

Step 3 $x + 9 = 4y$ Add 9 to each side.

$\frac{x + 9}{4} = y$ Divide each side by 4.

▶ The inverse of f is $g(x) = \frac{x + 9}{4}$, or $g(x) = \frac{1}{4}x + \frac{9}{4}$.

Method 2 Use inverse operations in the reverse order.

$f(x) = 4x - 9$ Multiply the input x by 4 and then subtract 9.

To find the inverse, apply inverse operations in the reverse order.

$g(x) = \frac{x + 9}{4}$ Add 9 to the input x and then divide by 4.

▶ The inverse of f is $g(x) = \frac{x + 9}{4}$, or $g(x) = \frac{1}{4}x + \frac{9}{4}$.

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Find the inverse of the function. Then graph the function and its inverse.

6. $f(x) = 6x$

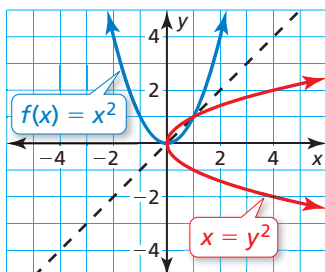
7. $f(x) = -x + 5$

8. $f(x) = \frac{1}{4}x - 1$

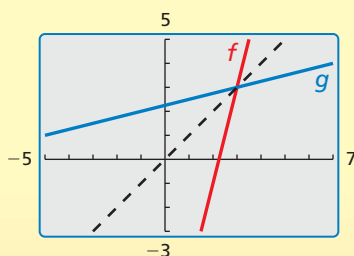
Finding Inverses of Nonlinear Functions

The inverse of the linear function in Example 3 is also a function. The inverse of a function, however, is *not* always a function. The graph of $f(x) = x^2$ is shown along with its reflection in the line $y = x$. Notice that the graph of the inverse of $f(x) = x^2$ does not pass the Vertical Line Test. So, the inverse is *not* a function.

When the domain of $f(x) = x^2$ is *restricted* to only nonnegative real numbers, the inverse of f is a function, as shown in the next example.



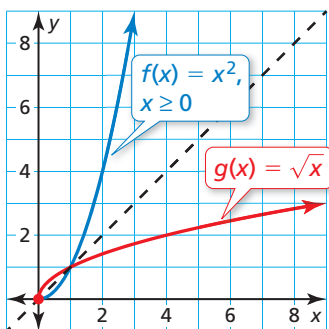
Check



The graph of g appears to be a reflection of the graph of f in the line $y = x$. ✓

EXAMPLE 4 Finding the Inverse of a Quadratic Function

Find the inverse of $f(x) = x^2, x \geq 0$. Then graph the function and its inverse.



SOLUTION

$$f(x) = x^2$$

Write the function.

$$y = x^2$$

Set y equal to $f(x)$.

$$x = y^2$$

Switch x and y in the equation.

$$\pm\sqrt{x} = y$$

Take square root of each side.

- Because the domain of f is restricted to nonnegative values of x , the range of the inverse must also be restricted to nonnegative values. So, the inverse of f is $g(x) = \sqrt{x}$.

You can use the graph of a function f to determine whether the inverse of f is a function by applying the *Horizontal Line Test*.

Core Concept

Horizontal Line Test

The inverse of a function f is also a function if and only if no horizontal line intersects the graph of f more than once.

EXAMPLE 5 Finding the Inverse of a Radical Function

Consider the function $f(x) = \sqrt{x+2}$. Determine whether the inverse of f is a function. Then find the inverse.

SOLUTION

Graph the function f . Because no horizontal line intersects the graph more than once, the inverse of f is a function. Find the inverse.

$$y = \sqrt{x+2}$$

Set y equal to $f(x)$.

$$x = \sqrt{y+2}$$

Switch x and y in the equation.

$$x^2 = (\sqrt{y+2})^2$$

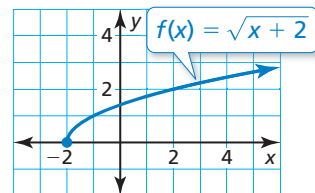
Square each side.

$$x^2 = y + 2$$

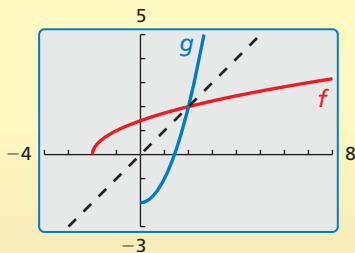
Simplify.

$$x^2 - 2 = y$$

Subtract 2 from each side.



Check



- Because the range of f is $y \geq 0$, the domain of the inverse must be restricted to $x \geq 0$. So, the inverse of f is $g(x) = x^2 - 2$, where $x \geq 0$.

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Find the inverse of the function. Then graph the function and its inverse.

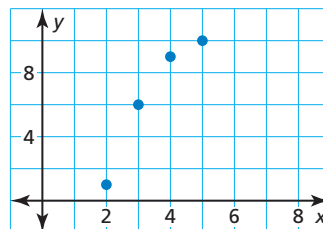
9. $f(x) = -x^2, x \leq 0$

10. $f(x) = 4x^2 + 3, x \geq 0$

11. Is the inverse of $f(x) = \sqrt{2x-1}$ a function? Find the inverse.

Vocabulary and Core Concept Check

- COMPLETE THE SENTENCE** A relation contains the point $(-3, 10)$. The _____ contains the point $(10, -3)$.
- DIFFERENT WORDS, SAME QUESTION** Consider the function f represented by the graph. Which is different? Find “both” answers.



Graph the inverse of the function.

Reflect the graph of the function in the x -axis.

Reflect the graph of the function in the line $y = x$.

Switch the inputs and outputs of the function and graph the resulting function.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the inverse of the relation.
(See Example 1.)

- $(1, 0), (3, -8), (4, -3), (7, -5), (9, -1)$
- $(2, 1), (4, -3), (6, 7), (8, 1), (10, -4)$

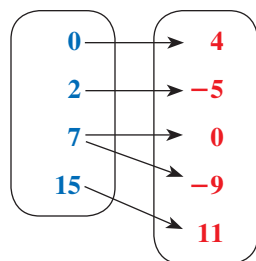
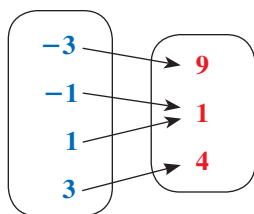
5.

Input	-5	-5	0	5	10
Output	8	6	0	6	8

6.

Input	-12	-8	-5	-3	-2
Output	2	5	-1	10	-2

7. **Input** **Output** 8. **Input** **Output**

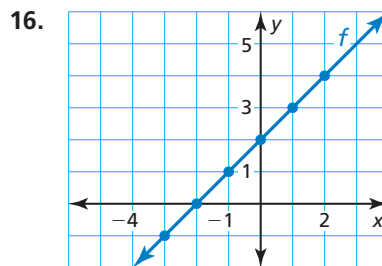
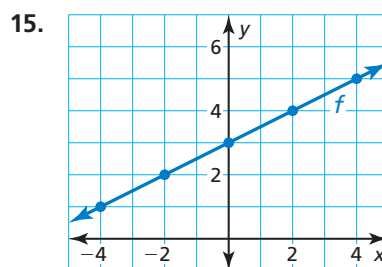


In Exercises 9–14, solve $y = f(x)$ for x . Then find the input when the output is 2. (See Example 2.)

- $f(x) = x + 5$
- $f(x) = 2x - 3$
- $f(x) = \frac{1}{4}x - 1$
- $f(x) = \frac{2}{3}x + 4$

- $f(x) = 9x^2$
- $f(x) = \frac{1}{2}x^2 - 7$

In Exercises 15 and 16, graph the inverse of the function by reflecting the graph in the line $y = x$. Describe the domain and range of the inverse.



In Exercises 17–22, find the inverse of the function. Then graph the function and its inverse. (See Example 3.)

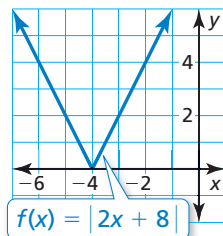
- $f(x) = 4x - 1$
- $f(x) = -2x + 5$
- $f(x) = -3x - 2$
- $f(x) = 2x + 3$
- $f(x) = \frac{1}{3}x + 8$
- $f(x) = -\frac{3}{2}x + \frac{7}{2}$

In Exercises 23–28, find the inverse of the function. Then graph the function and its inverse. (See Example 4.)

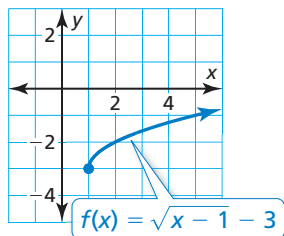
23. $f(x) = 4x^2, x \geq 0$ 24. $f(x) = -\frac{1}{25}x^2, x \leq 0$
25. $f(x) = -x^2 + 10, x \leq 0$
26. $f(x) = 2x^2 + 6, x \geq 0$
27. $f(x) = \frac{1}{9}x^2 + 2, x \geq 0$ 28. $f(x) = -4x^2 - 8, x \leq 0$

In Exercises 29–32, use the Horizontal Line Test to determine whether the inverse of f is a function.

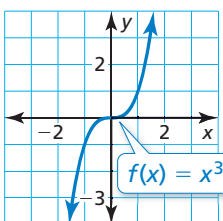
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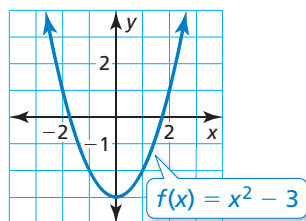
30.



31.



32.



In Exercises 33–42, determine whether the inverse of f is a function. Then find the inverse. (See Example 5.)

33. $f(x) = \sqrt{x + 3}$ 34. $f(x) = \sqrt{x - 5}$
35. $f(x) = \sqrt{2x - 6}$ 36. $f(x) = \sqrt{4x + 1}$
37. $f(x) = 3\sqrt{x - 8}$ 38. $f(x) = -\frac{1}{4}\sqrt{5x + 2}$
39. $f(x) = -\sqrt{3x + 5} - 2$
40. $f(x) = 2\sqrt{x - 7} + 6$
41. $f(x) = 2x^2$ 42. $f(x) = |x|$

43. **ERROR ANALYSIS** Describe and correct the error in finding the inverse of the function $f(x) = 3x + 5$.

X

$$y = 3x + 5$$

$$y - 5 = 3x$$

$$\frac{y - 5}{3} = x$$

The inverse of f is $g(x) = \frac{y - 5}{3}$, or $g(x) = \frac{y}{3} - \frac{5}{3}$.

44. **ERROR ANALYSIS** Describe and correct the error in finding and graphing the inverse of the function $f(x) = \sqrt{x - 3}$.

X

$$y = \sqrt{x - 3}$$

$$x = \sqrt{y - 3}$$

$$x^2 = y - 3$$

$$x^2 + 3 = y$$

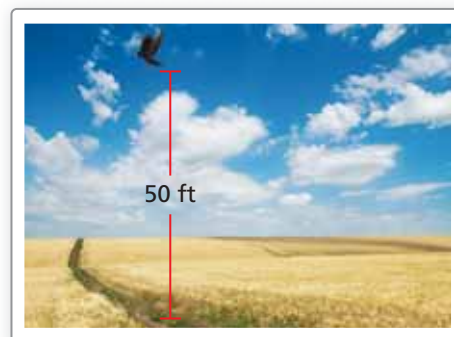
The inverse of f is $g(x) = x^2 + 3$.

45. **MODELING WITH MATHEMATICS** The euro is the unit of currency for the European Union. On a certain day, the number E of euros that could be obtained for D U.S. dollars was represented by the formula shown.

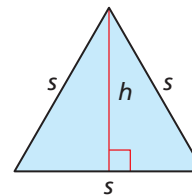
$$E = 0.74683D$$

Solve the formula for D . Then find the number of U.S. dollars that could be obtained for 250 euros on that day.

46. **MODELING WITH MATHEMATICS** A crow is flying at a height of 50 feet when it drops a walnut to break it open. The height h (in feet) of the walnut above ground can be modeled by $h = -16t^2 + 50$, where t is the time (in seconds) since the crow dropped the walnut. Solve the equation for t . After how many seconds will the walnut be 15 feet above the ground?



MATHEMATICAL CONNECTIONS In Exercises 47 and 48, s is the side length of an equilateral triangle. Solve the formula for s . Then evaluate the new formula for the given value.



47. Height: $h = \frac{\sqrt{3}s}{2}$; $h = 16$ in.
48. Area: $A = \frac{\sqrt{3}s^2}{4}$; $A = 11$ ft²

In Exercises 49–54, find the inverse of the function. Then graph the function and its inverse.

49. $f(x) = 2x^3$

50. $f(x) = x^3 - 4$

51. $f(x) = (x - 5)^3$

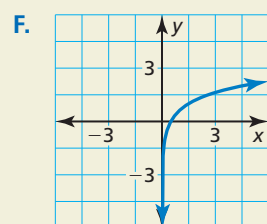
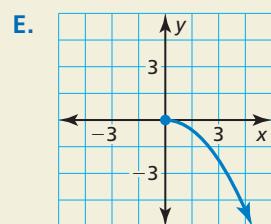
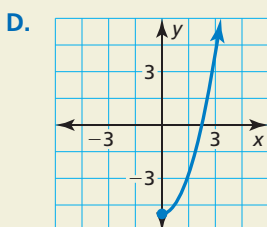
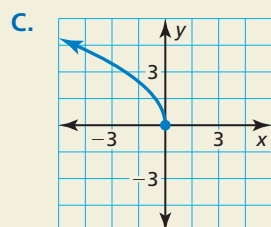
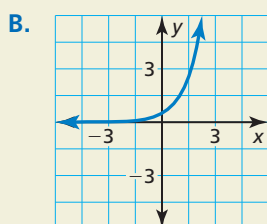
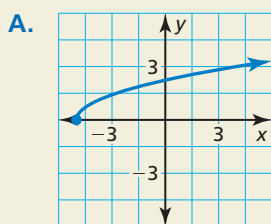
52. $f(x) = 8(x + 2)^3$

53. $f(x) = 4\sqrt[3]{x}$

54. $f(x) = -\sqrt[3]{x - 1}$

55. **MAKING AN ARGUMENT** Your friend says that the inverse of the function $f(x) = 3$ is a function because all linear functions pass the Horizontal Line Test. Is your friend correct? Explain.

56. **HOW DO YOU SEE IT?** Pair the graph of each function with the graph of its inverse.



57. **WRITING** Describe changes you could make to the function $f(x) = x^2 - 5$ so that its inverse is a function. Describe the domain and range of the new function and its inverse.

58. **CRITICAL THINKING** Can an even function with at least two values in its domain have an inverse that is a function? Explain.

59. **OPEN-ENDED** Write a function such that the graph of its inverse is a line with a slope of 4.

60. **CRITICAL THINKING** Consider the function $g(x) = -x$.

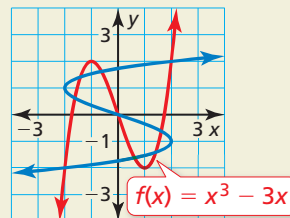
a. Graph $g(x) = -x$ and explain why it is its own inverse.

b. Graph other linear functions that are their own inverses. Write equations of the lines you graph.

c. Use your results from part (b) to write a general equation that describes the family of linear functions that are their own inverses.

61. **REASONING** Show that the inverse of any linear function $f(x) = mx + b$, where $m \neq 0$, is also a linear function. Write the slope and y-intercept of the graph of the inverse in terms of m and b .

62. **THOUGHT PROVOKING** The graphs of $f(x) = x^3 - 3x$ and its inverse are shown. Find the greatest interval $-a \leq x \leq a$ for which the inverse of f is a function. Write an equation of the inverse function.



63. **REASONING** Is the inverse of $f(x) = 2|x + 1|$ a function? Are there any values of a , h , and k for which the inverse of $f(x) = a|x - h| + k$ is a function? Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the sum or difference. (Section 7.1)

64. $(2x - 9) - (6x + 5)$

65. $(8y + 1) + (-y - 12)$

66. $(t^2 - 4t - 4) + (7t^2 + 12t + 3)$

67. $(-3d^2 + 10d - 8) - (7d^2 - d - 6)$

Graph the function. Compare the graph to the graph of $f(x) = x^2$. (Section 8.2)

68. $g(x) = x^2 + 6$

69. $h(x) = -x^2 - 2$

70. $p(x) = -4x^2 + 5$

71. $q(x) = \frac{1}{3}x^2 - 1$